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# Bounds for Jacobi and Related Polynomials Derivable by Matrix Methods*

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For a given matrix with dominant diagonal, a number of convenient upper and lower bounds for the corresponding determinant are available in the literature. These inequalities are used here to derive bounds for a general system of orthogonal polynomials for a specified range of values of the argument. The results thus obtained are applied to the classical Jacobi and related polynomials.

## 1. INTRODUCTION

Let $A = [a_{ij}]$ be a square matrix of complex numbers. It is said to have (strictly) dominant diagonal if

$$\forall k \left\{ | a_{kk} | > \sum_{i \neq k} | a_{ki} | = R_k \right\}, \tag{1}$$

i.e., the diagonal element dominates in each row. Thus, if a matrix has dominant diagonal,

$$\sigma_k \, | a_{kk} | = R_k , \tag{2}$$

with $0 \leqslant \sigma_k < 1$.

We recall the following theorems, due to Price [4] and Brenner [1], respectively, which give bounds on det $A$ when the elements $a_{ij}$ satisfy (1).

THEOREM A. *Let $A$ be an $n \times n$ matrix of complex numbers and let* (1) *hold.*

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*Then*

$$\prod_{k=1}^{n} L_k \leqslant |\det A| \leqslant \prod_{k=1}^{n} U_k, \tag{3}$$

*where*

$$\begin{cases} L_k = |a_{kk}| - \sum_{j>k} |a_{kj}|, \\ U_k = |a_{kk}| + \sum_{j>k} |a_{kj}|, \qquad k = 1,\dots, n. \end{cases} \tag{4}$$

THEOREM B. *Let $A$ be an $n \times n$ matrix of complex numbers and let* (2) *hold.*

*Then*

$$\prod_{k=1}^{n} L_k' \leqslant |\det A| \leqslant \prod_{k=1}^{n} U_k', \tag{5}$$

*where*

$$\begin{cases} L_k' = |a_{kk}| - \sum_{j>k} \sigma_j |a_{kj}|, \\ U_k' = |a_{kk}| + \sum_{j>k} \sigma_j |a_{kj}|, \qquad k = 1,\dots, n. \end{cases} \tag{6}$$

Making use of these theorems, Brenner [2] has given bounds for Legendre, Tchebichef, Laguerre, and Hermite polynomials. In this paper we first show that the results of Brenner [2] can appropriately be extended to hold for a general system of orthogonal polynomials. We then apply this general result to obtain bounds for the classical Jacobi polynomials defined by [5, p. 68]

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}, \qquad n = 0, 1, 2,\dots, \tag{7}$$

with $P_{-1}^{(\alpha,\beta)}(x) = 0$.

By specializing the parameters, $P_n^{(\alpha,\beta)}(x)$ can be reduced to Gegenbauer, Legendre, and Tchebichef polynomials; and by certain limiting processes, to Hermite, Laguerre and other polynomials. Thus it would seem fairly simple to deduce bounds for a large number of polynomial systems from the results given in this paper.

## 2. THE GENERAL RESULTS

Let $\{p_n(x)\}$ be a sequence of polynomials, where $p_n(x)$ is of degree precisely $n$ in $x$, orthogonal with respect to the weight function $w(x) > 0$ over the interval $(a, b)$. Then it is easy to show that the polynomial system $\{p_n(x)\}$ satisfies a three-term recurrence relation

$$p_{n+1}(x) = (A_n x + B_n)\, p_n(x) - C_n p_{n-1}(x), \qquad n = 0, 1, 2,..., \tag{8}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$, and $A_n$, $B_n$, $C_n$ are given in Ref. [3, Eq. (8), p. 159].

The recursion formula (8) enables us to write the determinantal forms

$$p_{n+1} = \det M_{n+1} \tag{9}$$

and

$$p_{n+1} = \det N_{n+1}, \tag{10}$$

for integers $n \geqslant 1$. Here $M_{n+1}$ is the $(n+1) \times (n+1)$ matrix with main diagonal $[A_n x + B_n ,..., A_0 x + B_0]$, superdiagonal $[C_n ,..., C_1]$, and subdiagonal $[1,...,1]$; $N_{n+1}$ is the $(n+1) \times (n+1)$ matrix with the same main diagonal, and with super- and subdiagonals both equal to $[\sqrt{C_n} ,..., \sqrt{C_1}]$. (Other elements are 0.)

Applying Theorem A to (9) we are led at once to

THEOREM 1. *For all $x$ such that*

$$\begin{cases} | A_0 x + B_0 | > 1, \\ | A_j x + B_j | > 1 + | C_j |, \qquad j = 1,..., n-1, \\ | A_n x + B_n | > | C_n |, \end{cases} \tag{11}$$

*the polynomial $p_{n+1}(x)$ satisfies the inequalities*

$$| A_0 x + B_0 | \prod_{j=1}^{n} \{| A_j x + B_j | - | C_j |\} \leqslant | p_{n+1}(x)|$$

$$\leqslant | A_0 x + B_0 | \prod_{j=1}^{n} \{| A_j x + B_j | + | C_j |\}, \qquad n \geqslant 1. \tag{12}$$

On the other hand, Theorem A when applied to (10) will yield a slight improvement over Theorem 1 given by

THEOREM 2. *For all $x$ such that*

$$\begin{cases} | A_0 x + B_0 | > | \sqrt{C_1} | \\ | A_j x + B_j | > | \sqrt{C_j} | + | \sqrt{C_{j+1}} |, \qquad j = 1,..., n-1, \\ | A_n x + B_n | > | \sqrt{C_n} |, \end{cases} \tag{13}$$

*the polynomial* $p_{n+1}(x)$ *satisfies the inequalities*

$$|A_0x+B_0|\prod_{j=1}^{n}\{|A_jx+B_j|-|\sqrt{C_j}|\}\leqslant|p_{n+1}(x)|$$
$$\leqslant|A_0x+B_0|\prod_{j=1}^{n}\{|A_jx+B_j|+|\sqrt{C_j}|\},\qquad n\geqslant 1. \tag{14}$$

It may be of interest to note that even better bounds for the polynomial $p_{n+1}(x)$, $n \geqslant 1$, are obtainable by using Theorem B instead of Theorem A. The details involved are straightforward and may, therefore, be omitted.

## 3. Bounds for Jacobi Polynomials

In this section we apply Theorems 1 and 2 to obtain upper and lower bounds for the classical Jacobi polynomials defined by (7). Indeed we know that the polynomial system $\{P_n^{(\alpha,\beta)}(x)\}$ satisfies the three-term recurrence formula (8) with $p_n(x) = P_n^{(\alpha,\beta)}(x)$, and

$$\begin{cases} D_nA_n = (2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2), \\ D_nB_n = (\alpha^2-\beta^2)(2n+\alpha+\beta+1), \\ D_nC_n = 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2), \end{cases} \tag{15}$$

where $D_n = 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)$.

With $A_n$, $B_n$, $C_n$ defined by (15), the inequalities in (11) are satisfied when $|x| \geqslant R + S_1$, and those in (13) when $|x| \geqslant R + S_2$, where

$$R = \max_{0\leqslant j\leqslant n}\left|\frac{\beta^2-\alpha^2}{(2j+\alpha+\beta)(2j+\alpha+\beta+2)}\right| \tag{16}$$

$$S_1 = \max_{0\leqslant j\leqslant n-1}\left|\frac{2(j+1)(j+\alpha+\beta+1)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta+2)}\right| + \max_{1\leqslant j\leqslant n}\left|\frac{2(j+\alpha)(j+\beta)}{(2j+\alpha+\beta)(2j+\alpha+\beta+1)}\right|, \tag{17}$$

and

$$S_2 = \max_{0\leqslant j\leqslant n-1}\left|\frac{2(j+1)(j+\alpha+\beta+1)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta+2)} \times\left[\frac{(j+\alpha+1)(j+\beta+1)(2j+\alpha+\beta+4)}{(j+2)(j+\alpha+\beta+2)(2j+\alpha+\beta+2)}\right]^{1/2}\right| + \max_{1\leqslant j\leqslant n}\left|\frac{2}{(2j+\alpha+\beta+1)} \times\left[\frac{(j+1)(j+\alpha)(j+\beta)(j+\alpha+\beta+1)}{(2j+\alpha+\beta)(2j+\alpha+\beta+2)}\right]^{1/2}\right|. \tag{18}$$

Thus we have the following consequences of Theorems 1 and 2:

THEOREM 1.1. *For* $n \geqslant 1$ *and* $|x| \geqslant R + S_1$, *the Jacobi polynomial* $P_{n+1}^{(\alpha,\beta)}(x)$ *satisfies the inequalities*

$$\prod_{j=1}^{n} \{|A_j x + B_j| - |C_j|\} \leqslant \tfrac{1}{2} |P_{n+1}^{(\alpha,\beta)}(x)| \, |(\alpha + \beta + 2)x + (\alpha - \beta)|^{-1} \leqslant \prod_{j=1}^{n} \{|A_j x + B_j| + |C_j|\}, \tag{19}$$

*where* $A_j$, $B_j$, $C_j$, $j = 1,..., n$, *are given by* (15).

THEOREM 2.1. *For* $n \geqslant 1$ *and* $|x| \geqslant R + S_2$, *the Jacobi polynomial* $P_{n+1}^{(\alpha,\beta)}(x)$ *satisfies the inequalities*

$$\prod_{j=1}^{n} \{|A_j x + B_j| - |\sqrt{C_j}|\} \leqslant \tfrac{1}{2} |P_{n+1}^{(\alpha,\beta)}(x)| \, |(\alpha + \beta + 2)x + (\alpha - \beta)|^{-1} \leqslant \prod_{j=1}^{n} \{|A_j x + B_j| + |\sqrt{C_j}|\}, \tag{20}$$

*where* $A_j$, $B_j$, $C_j$, $j = 1,..., n$, *are defined by* (15).

For suitable special values of the parameters $\alpha$ and $\beta$, the above theorems will yield bounds for the ultraspherical polynomial, the Legendre polynomial, the Tchebichef polynomials of the first and second kinds, and so on (cf. [5, p. 81, Eqs. (4.7.1) and (4.7.2); [3, pp. 183 and 191]). An interesting special case would arise if we suppose that $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$ and $\mathrm{Re}(\alpha + \beta) \geqslant 0$; then it is readily seen that $R = |(\beta - \alpha)/(\alpha + \beta + 2)|$, $S_1 \leqslant 3$ and $S_2 \leqslant (2 \, |\alpha + \beta + 4|)^{1/2}$, and we are thus led to the following results:

THEOREM 1.11. *For* $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$ *and* $\mathrm{Re}(\alpha + \beta) \geqslant 0$, *the inequalities in* (19) *hold when* $n \geqslant 1$ *and*

$$|x| \geqslant 3 + \left| \frac{\beta - \alpha}{\alpha + \beta + 2} \right|. \tag{21}$$

Indeed, this last condition (21) is satisfied for all admissible values of $\alpha$ and $\beta$ if, for instance, $|x| \geqslant 4$.

THEOREM 2.11. *For* $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$, and $\mathrm{Re}(\alpha + \beta) \geqslant 0$, *the inequalities in* (20) *hold when* $n \geqslant 1$ *and*

$$|x| \geqslant (2 \, |\alpha + \beta + 4|)^{1/2} + \left| \frac{\beta - \alpha}{\alpha + \beta + 2} \right|. \tag{22}$$

Evidently when $\alpha = \beta = \nu - \frac{1}{2}$ and $\mathrm{Re}(\nu) \geqslant \frac{1}{2}$, Theorems 1.11 and 2.11 will give upper and lower bounds for the Gegenbauer (or ultraspherical) polynomial $C_{n+1}^{\nu}(x)$, $n \geqslant 1$, for $|x| \geqslant 3$ and $|x| \geqslant (2|2\nu + 3|)^{1/2}$, respectively.

## References

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